PT-invariance and representations of the Temperley-Lieb algebra on the unit circle

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Abstract

We present in detail a recent conjecture on self-adjoint representations of the Temperley-Lieb algebra for particular values on the unit circle. The formulation in terms of graphical calculus is emphasized and discussed for several examples. The role of PT (parity and time reversal) invariance is highlighted as it might prove important for generalizing the construction to other cases.

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1 Introduction

We summarize recent novel results [1, 2] on the PT-invariant construction of self-adjoint representations of the Temperley-Lieb algebra [3]. The latter is defined as follows.

Definition 1 *Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q. The Temperley-Lieb algebra $TL_N(q)$ is the associative algebra over $\mathbb{C}(q)$ generated by $\{e_1, ..., e_{N-1}\}$ subject to the relations*

$$e_i^2 = -(q + q^{-1})e_i, \qquad e_i e_{i\pm 1} e_i = e_i, \qquad e_i e_j = e_j e_i \quad \text{for} \quad |i - j| > 1 \ . \tag{1}$$

It is well-kown that if the indeterminate is evaluated in the real numbers, $q \in \mathbb{R}$, or takes the special root-of-unity values

$$q = \exp(i\pi/r), \qquad r = 3, 4, 5, ... \ , \tag{2}$$

that there exist self-adjoint representations of $TL_N(q)$ [4, 6, 7, 5].

The novel aspect we wish to highlight here are recent results [1] of the construction of self-adjoint representations for the above values on the unit circle using ideas from quasi-Hermitian quantum mechanics and PT-invariance, e.g. [8, 9, 10, 11]. This approach is physically motivated: it is a *preparatory* step to explicitly construct for *arbitrary* values $q \in \mathbb{S}^1$ representations of $TL_N(q)$ for which the following quantum Hamiltonian

$$H = \sum_{i=1}^{N-1} e_i \tag{3}$$

is Hermitian or self-adjoint. The latter requirement is necessary in order to allow for a sound physical interpretation of the associated quantum model. Clearly, the case when each of the Temperley-Lieb generators e_i is self-adjoint is a particular subset of this class of representations.

The case when q is evaluated on the unit circle is of special physical interest. For instance, at the aforementioned values (2) the model is related to the Q-state Potts model with $Q = (q+q^{-1})^2$. For the value $q = \exp(i\pi/2)$ the Hamiltonian (3) describes critical dense polymers on the square lattice, while for $q = \exp(2\pi i/3)$ it is related to the problem of percolation. More generally, it has been argued[12, 13] that the Hamiltonian (3) can be viewed as a discrete system which either in the thermodynamic limit ($N \to \infty$) or through its algebraic properties can be effectively described by logarithmic conformal field theories. While these applications are beyond the scope of this article, it needs to be stressed that they provide some of the basic motivation for the present construction using quasi-Hermiticity and PT-invariance.

As we wish to outline the basic principles of the approach we shall focus on the special case

$$q = \exp(i\pi/r), \qquad r > N, \tag{4}$$

where N is the number of strands and r can take *any real* values greater than N (not only integer values). This section of the unit circle - while not of immediate physical interest as it shrinks to $q = 1$ as $N \to \infty$ - is distinguished mathematically as it not only allows for a self-adjoint representation of the Temperley-Lieb algebra but also for the application of graphical calculus in terms of Kauffman diagrams [14]. This graphical formulation of the Temperley-Lieb algebra is very elegant and greatly facilitates computations. For these reasons we wish to maintain it when $q \in \mathbb{S}^1$. In this article we shall describe in more detail a construction of an inner product [2] which achieves this for the values (4). Before we can start our discussion we need to recall some previous results on PT-invariance and self-adjoint representations of the Temperley-Lieb algebra.

2 Review of previous results

We will concentrate on the $U_q(sl_2)$-invariant XXZ quantum spin-chain model [15, 16]. The latter model corresponds to a realisation of the Hamiltonian (3) in terms of the fundamental two-dimensional $U_q(sl_2)$-module. Prior to introducing it, we recall the following definition.

Definition 2 *The q-deformed enveloping algebra (or quantum group) $U_q(sl_2)$ is the associative algebra over $\mathbb{C}(q)$ generated by $\{E, F, K, K^{-1}\}$ subject to the relations*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ . \tag{5}$$

$U_q(sl_2)$ can be endowed with structure of an Hopf algebra with co-multiplication

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \tag{6}$$

and co-unit

$$\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K^{\pm 1}) = 1 \ . \tag{7}$$

There is also an antipode but we will not use it in the following.

Setting

$$V = \mathbb{C}v_+ \oplus \mathbb{C}v_- \tag{8}$$

we define the two-dimensional fundamental $U_q(sl_2)$-module by

$$Ev_+ = 0, \qquad Ev_- = v_+, \qquad Fv_- = 0, \qquad Fv_+ = v_-, \qquad Kv_\pm = q^{\pm 1}v_\pm \ . \tag{9}$$

There is a natural inner product on V given by

$$\langle v_\sigma, v_{\sigma'} \rangle = \delta_{\sigma,\sigma'}, \qquad \sigma, \sigma' = \pm 1 \ . \tag{10}$$

We choose the inner product to be antilinear in the first factor. Consider now the N-fold tensor product $V^{\otimes N}$ of the fundamental representation with the inner product

$$\langle v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N}, v_{\sigma'_1} \otimes \cdots \otimes v_{\sigma'_N} \rangle = \delta_{\sigma_1,\sigma'_1} \cdots \delta_{\sigma_N,\sigma'_N} \ . \tag{11}$$

Then the Temperley-Lieb algebra has the following matrix representation over $V^{\otimes N}$,

$$e_i \mapsto \underbrace{1 \otimes \cdots 1}_{i-1} \otimes \boldsymbol{e} \otimes \underbrace{1 \cdots \otimes 1}_{N-i-1}, \tag{12}$$

where the matrix elements $\boldsymbol{e}_{\sigma,\sigma'} := \langle v_\sigma, \boldsymbol{e} v_{\sigma'} \rangle$ of the operator $\boldsymbol{e} : V \otimes V \to V \otimes V$ are

$$(\boldsymbol{e}_{\sigma,\sigma'}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{13}$$

Note that with respect to the Hamiltonian (3) the quantum group provides a symmetry,

$$[H, U_q(sl_2)] = 0 \ .$$

This is a direct consequence of the quantum analogue of Schur-Weyl duality [18].

In the following we will now evaluate q to be a complex number. If q is real then one finds that the above product is invariant with respect to the action of the Temperley-Lieb algebra,

$$q \in \mathbb{R} : \qquad \langle v, e_i w \rangle = \langle e_i v, w \rangle, \quad v, w \in V^{\otimes N} \ . \tag{14}$$

For the quantum group generators one finds

$$q \in \mathbb{R}: \qquad \langle v, \Delta^{(N)}(x) w\rangle = \langle \Delta^{(N)}_{op}(x) v, w\rangle, \quad v,w \in V^{\otimes N},\ x \in U_q(sl_2), \tag{15}$$

where $\Delta_{op} = \tau \circ \Delta$ is the opposite coproduct with τ being the "flip"-operator, $\tau(x \otimes y) = y \otimes x$, and

$$\Delta^{(N)} = (1 \otimes \Delta)\Delta^{(N-1)} = (\Delta \otimes 1)\Delta^{(N-1)}, \qquad \Delta^{(2)} \equiv \Delta\ . \tag{16}$$

The opposite coproduct $\Delta^{(N)}_{op}$ is defined analogously.

In contrast, if $q \neq \pm 1$ lies on the unit circle $\mathbb{S}^1$ the inner product is no longer invariant,

$$q \in \mathbb{S}^1,\ q \neq \pm 1: \qquad \langle v, e_i w\rangle \neq \langle e_i v, w\rangle, \quad v,w \in V^{\otimes N} \tag{17}$$

and the Hamiltonian (3) ceases to be Hermitian. On physical grounds we therefore need to introduce a new inner product which renders H Hermitian. For the values (2) and (4) it turns out that this new inner product can also be chosen to be invariant with respect to the Temperley-Lieb action [1]. The language which we are going to employ in the construction of the invariant product is physically motivated, but as we will see the associated concepts have a clear mathematical interpretation and can be generalized beyond the section (4) of the unit circle.

2.1 Quasi-Hermiticity and PT-invariance

We wish to construct a map $\eta : V^{\otimes N} \to V^{\otimes N}$ which has the following properties:

1. It is Hermitian, $\langle v, \eta w\rangle = \langle \eta v, w\rangle$, invertible, $\det \eta \neq 0$, and positive, $\eta > 0$.
2. It intertwines the Hamiltonian and its Hermitian adjoint with respect to the inner product (11),
$$\eta H = H^* \eta\ . \tag{18}$$

The properties listed uner (1) guarantee that the *new* inner product $\langle \cdot\,,\cdot\,\rangle_\eta : V^{\otimes N} \times V^{\otimes N} \to \mathbb{C}$ given by

$$\langle v, w\rangle_\eta := \langle v, \eta w\rangle, \qquad v,w \in V^{\otimes N} \tag{19}$$

is well-defined, while property (2) ensures that the Hamiltonian becomes Hermitian,

$$\langle Hv, w\rangle_\eta = \langle v, Hw\rangle_\eta, \qquad v,w \in V^{\otimes N}\ . \tag{20}$$

Clearly, the existence of such a map η is not guaranteed but needs to be proved and - for all practical purposes - we wish to obtain η explicitly. This has been achieved [1] so far for (2) and (4) using ideas from quantum group reduction [17]. We omit the details here and instead will explicitly state the new inner product (and with it η) for (4) below, after introducing a convenient graphical formalism.

Besides the above requirements, which render the Hamiltonian H quasi-Hermitian, one can impose further constraints based on certain transformation properties in connection with parity, time and spin-reversal.

Definition 3 *Let P (parity-reversal), T (time-reversal) and R (spin-reversal) be the involutions $V^{\otimes N} \to V^{\otimes N}$ defined in terms of the following action on the basis elements*

$$\begin{aligned} P &: \quad v_{\sigma_1} \otimes v_{\sigma_2} \cdots \otimes v_{\sigma_N} \mapsto v_{\sigma_N} \otimes v_{\sigma_{N-1}} \cdots \otimes v_{\sigma_1}, \\ T &: \quad \alpha\ v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mapsto \bar{\alpha}\ v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N}, \quad \alpha \in \mathbb{C}, \\ R &: \quad v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mapsto v_{-\sigma_1} \otimes \cdots \otimes v_{-\sigma_N}\ . \end{aligned}$$

We define P, R to be linear but T to be anti-linear.

Remark. While the definition of the parity and spin-reversal operators is conceptually clear, the identification of the antilinear map T with time-reversal warrants an additional comment. One important method to construct eigenvectors of the quantum Hamiltonian (3) is the coordinate Bethe ansatz. The latter involves the definition of a discrete quantum mechanical wave function ψ. The map T is defined such that this wave function is transformed into its complex conjugate, $\psi \to \bar{\psi}$. This transformation of the wave function which ought to obey a discrete version of the Schrödinger equation, $i\partial_t\psi = H\psi$, corresponds to time-reversal.

A straightforward computation exploiting (12), (13) shows that the Hamiltonian (3) is PT and RT-invariant. Namely, we have that

$$\langle Hv, w\rangle = \langle v, PHPw\rangle = \langle v, RHRw\rangle = \langle v, THTw\rangle, \qquad v, w \in V^{\otimes N} \ . \tag{21}$$

These identities motivate the following additioanl requirements on the map η,

$$P\eta P = R\eta R = T\eta T = \eta^{-1} \ . \tag{22}$$

We shall refer to these transformations as PT and RT-invariance of the map η and its associated inner product, respectively. An immediate consequence of these relations is that

$$\det \eta = \det P\eta^{-1}P = \frac{1}{\det \eta} \ \Rightarrow \ \det \eta = 1 \ . \tag{23}$$

As already mentioned such a map η ensuring quasi-Hermiticity of the Hamiltonian and satisfying PT and RT-invariance does indeed exist and, moreover, can be explicitly constructed [1]. We summarize the previous results [1] in the following theorem.

Theorem 1 *Evaluate q in the section (4) of the unit circle. Then there exists a map $\eta : V^{\otimes N} \to V^{\otimes N}$ possessing the properties mentioned above and in addition enjoys the more restrictive constraints*

$$\langle e_i v, w\rangle_\eta = \langle v, e_i w\rangle_\eta, \qquad i = 1, 2, ..., N-1 \tag{24}$$

and

$$\langle \Delta_{op}^{(N)}(\varphi(x))v, w\rangle_\eta = \langle v, \Delta^{(N)}(x)w\rangle_\eta, \qquad x \in U_q(sl_2) \tag{25}$$

where $v, w \in V^{\otimes N}$ and φ is the $U_q(sl_2)$-automorphism

$$\varphi(K^{\pm 1}) = K^{\mp 1}, \quad \varphi(E) = F, \quad \varphi(F) = E, \quad \varphi(xy) = \varphi(y)\varphi(x), \ x, y \in U_q(sl_2) \ . \tag{26}$$

If one explicitly computes η using the previous results in the literature [1], one quickly realizes the importance of the choice of basis. For example, if η is to be computed with respect to the basis vectors

$$\{v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mid \sigma_i = \pm 1\} \subset V^{\otimes N} \tag{27}$$

one finds in general none of its matrix element are nonzero within a fixed spin sector, i.e. in a subspace with $\sum_i \sigma_i =$ const. This makes an evaluation of the inner product (19) and computations of physically relevant quantities extremely cumbersome. We therefore shall introduce another basis which is algebraically motivated: it transforms very simply under the action of the Temperley-Lieb algebra and the quantum group. Most importantly, in this basis the action of the latter two algebras can be described graphically.

3 Change of basis and graphical calculus

The new basis, we denote it by $\{t_i\}$, is not *orthonormal* and leads to a shift in emphasis from the map $\eta : V^{\otimes N} \to V^{\otimes N}$ discussed above to the Gram matrix

$$G_{ij} = \langle t_i, \eta t_j \rangle \ . \tag{28}$$

The latter nicely reflects the algebraic properties of the new basis and can be evaluated by graphical means. Crucial for this graphical evaluation is the fact that there is a correspondence between basis vectors t_i and elements a_i in the Temperley-Lieb algebra. Using this correspondence one can directly define the Gram matrix in terms of a real functional

$$\omega : TL_N(q) \to \mathbb{R} \tag{29}$$

by setting

$$G_{ij} := \omega(a_i^* a_j) \tag{30}$$

where $*$ denotes a conjugation in the Temperley–Lieb algebra which corresponds to taking the Hermitian adjoint in the associated representation. The values of the functional can be computed using Kauffman diagrams. This suggests to circumvent the construction of the map η in the spin basis (27) entirely and instead to focus on the algebraically distinguished Gram matrix G. Obviously, all the properties of η can be translated to properties of the matrix G and it is convenient to do so because many matrix elements of G turn out to be vanishing. We shall list the properties of the Gram matrix G below. First we introduce the new basis vectors.

3.1 The new basis in terms of Young tableaux

The new basis $\{t_i\}$ which are going to define is closely related to the dual canonical basis discussed by Frenkel and Khovanov [19], see also [20] and [21] and references therein. The alert reader will notice, however, that there are important differences in our conventions from the ones used by the latter authors, since we need to accommodate that q lies on the unit circle. In particular our definition of the inner product, respectively the Gram matrix G, differs from the one for q real where the construction of Frenkel and Khovanov applies.

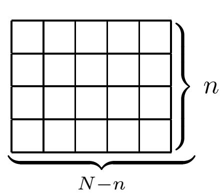
We start by decomposing the representation space $V^{\otimes N}$ with respect to the number of "down spins",

$$V^{\otimes N} = \bigoplus_{n=0}^{N} W_n, \qquad W_n = \operatorname*{span}_{\mathbb{C}} \{v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mid \textstyle\sum_i \sigma_i = N - 2n\} \ . \tag{31}$$

Note that the action of the Temperley-Lieb algebra respects this decomposition,

$$TL_N(q) W_n = W_n \ . \tag{32}$$

In each subspace W_n we now introduce the following basis [22]. Let λ_n be the rectangular Young diagram with n rows of $N - n$ boxes,

$\lambda_n =$ 

Then we assign to each subdiagram $\lambda' \subset \lambda_n$ a vector in W_n as follows. Let t be the unique standard tableau (column and row strict) of shape λ' whose entries are consecutive integers

with entry n in the upper left corner. For example,

$$t = \begin{array}{|c|c|c|c|c|} \hline n & n+1 & n+2 & \cdots & s \\ \hline n-1 & n & \cdots & s-2 \\ \cline{1-4} \vdots & & \\ \cline{1-3} s' \\ \cline{1-1} \end{array} \quad , \quad n < s < N, \quad 1 \leq s' < n\,. \tag{33}$$

Reading the entries of the tableau from left to right and top to bottom we set

$$t \mapsto e_{s'} e_{s'-1} \cdots e_{s-2} \cdots e_{n-1} e_s \cdots e_{n+1} e_n \Omega_n, \tag{34}$$

where

$$\Omega_n = \underbrace{v_- \otimes v_- \cdots \otimes v_-}_{n} \otimes v_+ \otimes v_+ \cdots \otimes v_+ \tag{35}$$

is the vector corresponding to $\lambda' = \varnothing$. Note that for fixed n there are as many of these tableaux as the dimension of the subspace W_n, namely $\dim W_n = \binom{N}{n}$.

Example. Let $N = 5$ and $n = 2$ then we have the following Young diagrams and tableaux:

$$t = \varnothing, \quad \begin{array}{|c|}\hline 2 \\ \hline\end{array}, \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline\end{array}.$$

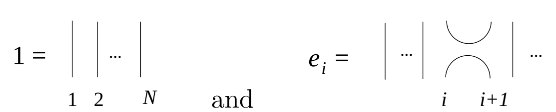
The corresponding algebra elements $a \in TL_N(q)$ are

$$a = 1,\ e_2,\ e_1e_2,\ e_3e_2,\ e_1e_3e_2,\ e_2e_1e_3e_2,\ e_4e_3e_2,\ e_1e_4e_3e_2,\ e_2e_1e_4e_3e_2,\ e_3e_2e_1e_4e_3e_2\ .$$

Note that we do not distinguish in our notation between Young tableaux and the associated basis vectors. Henceforth, we will also often identify the basis vectors respectively tableaux with the corresponding algebra elements.

3.2 Kauffman and oriented cup diagrams

There is an elegant graphical calculus connected with the new basis. Represent each algebra element in $TL_N(q)$ in terms of Kauffman diagrams, see the graphical depiction below,



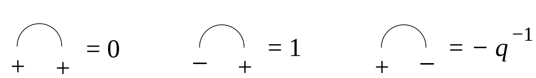
and realize the multiplication through concatenation from above. To compute the action of the algebra on the basis $\{t_i\}$ we can identify each basis vector with a half or cup-diagram which carries an orientation. Inspired by the formalism of Frenkel and Khovanov [19] we introduce the following graphical rules. Define a "cap" to be the map $\cap : V \otimes V \to \mathbb{C}$ with

$$v_+ \otimes v_+ \mapsto 0, \qquad v_+ \otimes v_- \mapsto -q^{-1}, \qquad v_- \otimes v_+ \mapsto 1, \qquad v_- \otimes v_- \mapsto 0 \tag{36}$$

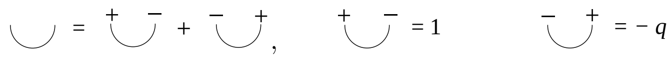
and a "cup" $\cup$ to be the map $\cup : \mathbb{C} \to V \otimes V$ such that

$$1 \mapsto v_+ \otimes v_- - q\ v_- \otimes v_+\ . \tag{37}$$

Graphically these maps are represented as follows



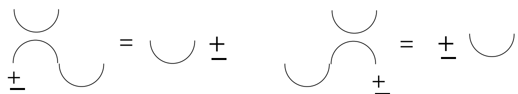
and



Each Temperley-Lieb generator $e_i : V_i \otimes V_{i+1} \to V_i \otimes V_{i+1}$ then corresponds to the composition

$$e_i = \cup_{i,i+1} \circ \cap_{i,i+1}, \tag{38}$$

where $V_i \otimes V_{i+1}$ are the i^{th} and $(i+1)^{\text{th}}$ copy in the tensor product $V^{\otimes N}$. Employing the graphical rules



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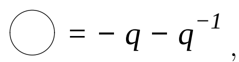


one can now easily generate the basis $\{t_i\}$ by acting with the corresponding algebra elements $\{a_i\}$ on the cyclic vector Ω_n. Applying the same graphical rules one also easily deduces that the action of $TL_N(q)$ simply permutes the basis elements $\{t_i\}$ up to factors of $-(q+q^{-1})$.

Besides a simple action of the Temperley-Lieb algebra the new basis vectors display also a nice transformation behaviour under the action of the quantum group [19]. Notice that the spin sectors are not preserved under the $U_q(sl_2)$-action, instead we have for the quantum group generators E, F that

$$E : W_n \to W_{n-1} \qquad \text{and} \qquad F : W_n \to W_{n+1}\ . \tag{39}$$

We now describe the action of E. Suppose we are given a cup diagram/Young tableaux t with k_- down spins (minus signs) then

$$E = \sum_{m=1}^{k_-} [m]_q E_m, \qquad [m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}, \tag{40}$$

where E_m connects the m^{th} and $(m+1)^{\text{th}}$ down spin with a cup (here any intermediate cups are ignored in the counting). For $m = k_-$ the map E_k simply flips the rightmost down-spin (minus sign) to an up-spin (plus sign). Note that by construction of the basis $\{t_i\}$ all down-spins are to the left of all up-spins. Similarly, the action of F on a cup diagram with k_+ up-spins can be described in terms of the sum

$$F = \sum_{m=1}^{k_+} [m]_q F_m, \tag{41}$$

where F_m connects the m^{th} and $(m+1)^{\text{th}}$ up-spin and one starts counting from the right. Again, any intermediate cups are ignored and F_{k_+} simply flips the leftmost up-spin to a down spin.

Example. Let $N = 5$ and take the tableau

$$t = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} = + \cup + + \ .$$

Then we have $k_+ = 3$ and

$$F\ \ + \cup + + = (+ \cup \cup) + [2]_q (\Cup +) + [3]_q (- \cup + +)\ .$$

Using the above graphical calculus one can now easily verify that the matrices defined through

$$e_i t_j = \sum_k t_k (\varepsilon_i)_{kj} \tag{42}$$

and

$$Et_j = \sum_k t_k \mathcal{E}_{kj} , \qquad Ft_j = \sum_k t_k \mathcal{F}_{kj} \tag{43}$$

are all real valued, but not symmetric. Because of (42) this particularly applies also to the matrix of the Hamiltonian (3),

$$Ht_j = \sum_k t_k \mathcal{H}_{kj} , \qquad \mathcal{H}_{kj} \in \mathbb{R} \qquad \text{and} \qquad \mathcal{H} \neq \mathcal{H}^t \; . \tag{44}$$

In contrast, the Hamiltonian matrix with respect to the spin basis (27) is symmetric but not real valued. Thus, as before we need to introduce a new inner product in terms of the Gram matrix (28) with respect to which the Hamiltonian becomes Hermitian.

4 Graphical definition of the Gram matrix

We are now ready to introduce the Gram matrix with respect to the new basis defined in the previous section. Since we wish to generalize the construction described here to other values of q on the unit circle in future work (see the comments in the introduction), it is worthwhile to first formulate its general properties before specializing to the section (4).

The analogues of property (1) and (2) for η are:

1. G is Hermitian, $G_{ij} = \bar{G}_{ji}$, invertible, $\det G \neq 0$, and positive, $G > 0$.
2. G intertwines the matrix $\mathcal{H}$ with its transpose,

$$G\mathcal{H} = \mathcal{H}^t G \; . \tag{45}$$

Besides these "minimal" requirements on G we can impose the additional constraints originating from PT and RT-invariance. Namely, from the equalities (22) we deduce that

$$G_{ij} = \langle Tt_i , \eta^{-1} Tt_j \rangle = \langle PTt_i , \eta PTt_j \rangle = \langle RTt_i , \eta RTt_j \rangle$$

and hence

$$\pi^* G \pi = G \qquad \text{and} \qquad \rho^* G \rho = G,$$

where

$$PT \; t_i = \sum_j t_j \pi_{ji} \qquad \text{and} \qquad RT \; t_i = \sum_j t_j \rho_{ji} \; .$$

Employing Hermiticity in conjunction with time-reversal, we find that the matrix

$$\eta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \langle v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} , \eta v_{\sigma'_1} \otimes \cdots \otimes v_{\sigma'_N} \rangle$$

obeys the identities $\eta^{-1} = \eta^t = \bar{\eta} = T\eta T$ from which we conclude that

$$G_{ij} = G_{ji} \in \mathbb{R} \; . \tag{46}$$

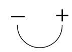
Finally, G also inherits the following properties from η: G is block-diagonal with respect to the decomposition (31) and in addition $\det G = 1$.

We now introduce for (4) a Gram matrix which satisfies all of these requirements and, furthermore, is invariant under the action of the Temperley-Lieb algebra. As already hinted at previously we define G in terms of a functional over the Temperley-Lieb algebra which can be computed in terms of Kauffman diagrams. This functional is the subject of the next definition.

Definition 4 *Identify each $a \in TL_N(q)$ with its Kauffman diagram and fix an integer $0 \leq n \leq N$. Assign to the top and bottom of the diagram the orientation*

$$\boldsymbol{\sigma}_n = \{\underbrace{-,...,-}_{n},\underbrace{+,...,+}_{N-n}\}\ .$$

Let x be the number of anti-clockwise oriented cups



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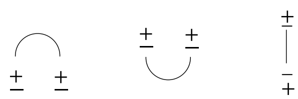
y the number of closed loops, and z the number of unoriented cups, caps or through lines,

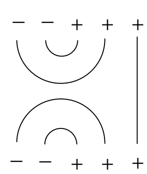


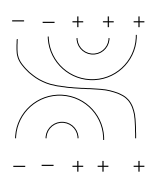
.

*Then we define the following functional $\omega_n : TL_N(q) \to \mathbb{R}$ by setting**

$$a \mapsto \omega_n(a) = \begin{cases} (-)^{x+y}(q+q^{-1})^y \dfrac{q^{\frac{N}{2}-n}+q^{n-\frac{N}{2}}}{q^{\frac{N}{2}-x}+q^{x-\frac{N}{2}}}, & \text{if } z=0 \\ 0, & \text{else} \end{cases}\ .$$

Example. We illustrate the above definition for two examples. Let $N = 5,\ n = 2$ and set $a = e_1e_2e_3e_2$ and $b = e_3e_2e_1e_4e_3e_2$. Then the associated oriented Kauffman diagrams are



and


.

We thus obtain

$$\omega_{n=2}(a) = 1 \qquad \text{and} \qquad \omega_{n=2}(b) = 0\ .$$

Conjecture 2 *Let $\{t_i\}$ denote the basis of $V^{\otimes N}$ described above in terms of Young tableaux and $\{a_i\} \subset TL_N(q)$ be the corresponding algebra elements. For each $n = 0, 1, 2, ..., N$ we set*

$$G_{ij} = \langle t_i, \eta t_j \rangle := \omega_n(a_i^* a_j), \qquad \forall t_i,\ t_j \in W_n\ . \tag{47}$$

Here $* : TL_N(q) \to TL_N(q)$ is the antilinear automorphism defined by

$$(e_{i_1}e_{i_2}\cdots e_{i_k})^* = e_{i_k}e_{i_{k-1}}\cdots e_{i_1} \tag{48}$$

and in terms of Kauffman diagrams is realized by flipping at the horizontal axis. The matrix G satisfies all of the above identities, in particular those arising from PT-invariance, and in addition obeys the relations

$$G\mathcal{E} = \mathcal{F}^t G \qquad \text{and} \qquad G\varepsilon_i = \varepsilon_i^t G, \qquad i = 1, ..., N-1, \tag{49}$$

where $\mathcal{E}, \mathcal{F}$ and ε_i are the matrix expressions for the quantum group and Temperley-Lieb generators in the basis $\{t_i\}$ as introduced earlier, see (43) and (42).

*Previously, we distinguished the case N odd and even [2]. However, simplifying the expression for N odd one can see that both cases are described by the same formula.

Remark. The implicit definition of the map η contained in the above expression for the Gram matrix G coincides with the earlier construction [1]. In fact, this is one way of checking the above conjecture [2]. Alternatively, one can verify the various identities and properties independently of the map η and this the point of view which we have taken here. Numerical checks of the above conjecture have been carried out for $N = 2, 3, 4, 5, 6, 7, 8$.

Note that there are simplifications in the computation of the inner product if we restrict to certain subspaces W' which are left invariant under the Temperley-Lieb action within a fixed sector W_n. Namely, set $n = \lfloor N/2 \rfloor$ (the integer part of $N/2$) and consider the subspace $W_{\max} \subset W_n$ of cup diagrams with a maximal number of cups. From the graphical calculus reviewed earlier, it is clear that

$$TL_N(q)W_{\max} = W_{\max}$$

and that for N even

$$EW_{\max} = FW_{\max} = \{0\} \ .$$

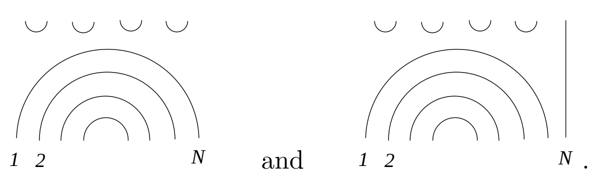
If N is odd we have obviously two subspaces $W_{\max}^{\pm}$ due to one unpaired vector or spin with

$$EW_{\max}^{-} = W_{\max}^{+} \qquad \text{and} \qquad FW_{\max}^{+} = W_{\max}^{-} \ .$$

In terms of Young tableaux these subspaces are spanned by all t which contain the "staircase" tableaux

$$\begin{array}{|c|c|c|c|c|}
\hline
n & n+1 & \cdots & \cdots & 2\left\lfloor \frac{N}{2} \right\rfloor - 1 \\
\hline
n-1 & n & \cdots & & \\
\cline{1-4}
\vdots & & & & \\
\cline{1-3}
2 & 1 & & & \\
\cline{1-2}
1 & & & & \\
\cline{1-1}
\end{array} \ .$$

The corresponding Kauffman diagrams for N even and odd look as follows,



From this graphical representation it is apparent that for each Kauffman diagram associated with the algebra element $a_i^* a_j,\ a_i, a_j \in W_{\max}$ the number x of anti-clockwise oriented cups equals the number n of down spins (minus signs). Thus, we may for this special case compute the Gram matrix purely in terms of the upper half diagrams (= cup diagrams) as we only need to count the number of closed loops, i.e.

$$(G|_{W_{\max}})_{ij} = \omega_n(a_i^* a_j) = (-q - q^{-1})^{y_{ij}} ,$$

where $y_{ij} = y_{ji}$ is the number of closed loops in $a_i^* a_j$. We illustrate the comments just made for a simple example.

Example. Set $N = 7$ and $n = 3$. Then the subspace $W_{\max}$ of all half-diagrams containing 3 cups is spanned by the tableaux t which contain

$$\begin{array}{|c|c|c|}
\hline
3 & 4 & 5 \\
\hline
2 & 3 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array} \quad \equiv \cup \cup \cup + \quad .$$

Here we have expressed the corresponding basis vector

$$e_1e_3e_2e_5e_4e_3\ v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_+ \otimes v_+$$

in terms of a cup diagram by omitting the lower half diagram. If we now wish to compute the scalar product between the two diagrams

$$\begin{array}{|c|c|c|}\hline 3&4&5\\ \hline 2&3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \equiv \cup\cup\cup+ \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 3&4&5&6\\ \hline 2&3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \equiv \cup\cup+\cup$$

we simply need to combine them by flipping one of them at the horizontal axis and count the number of closed loops. We find

$$\langle \cup\cup\cup+, \cup\cup+\cup \rangle = (q+q^{-1})^2 \ .$$

Note, that this simplification of computing the Gram matrix purely in terms of half or cup-diagrams is not possible in general when computing scalar products between diagrams which differ in their number of cups. For instance, to determine the scalar product between a vector in W_n and the cyclic vector Ω_n requires the full diagram. The computation of these scalar products is necessary to ensure quasi-Hermiticity of the Hamiltonian on the entire state space $V^{\otimes N}$.

5 Discussion

The Temperley-Lieb algebra arises in the context of quantum integrable models whose dynamics and quantum statistics is described by the Hamiltonian (3) and we have discussed for a special example how to render it Hermitian by constructing an appropriate inner product. These quantum integrable models are closely related to classical two-dimensional statistical mechanics models which are defined in terms of the following solution of the quantum Yang-Baxter equation,

$$R_{i,i+1}(x) = \mathbf{1} + \frac{x-x^{-1}}{xq-x^{-1}q^{-1}}\ e_i \ . \tag{50}$$

There are various integrable boundary conditions one can impose on a square-lattice and here we only concentrated on those which turn the model quantum group invariant. In context of the six-vertex model solution the statistical transfer matrix then reads

$$t(x) = \operatorname*{Tr}_0 K_0 R_{M,0}(x)R_{M-1,M}(x)\cdots R_{1,2}(x)^2 R_{2,3}(x)\cdots R_{M-1,M}(x)R_{M,0}(x),$$

where the boundary conditions are encoded in the only non-trivial boundary matrix

$$K = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \ .$$

Obviously, one can widen the discussion to include the transfer matrix and in the cases (2) and (4) the latter turns out to be Hermitian as well.

In comparison, different choice of products have been made in the literature, e.g. in the context of applications to logarithmic conformal field theory the choice $G_{ij} = \delta_{ij}$ leads to non-trivial Jordan blocks in the Hamiltonian and transfer matrix [12]. The results presented here show that other choices might be possible where the Hamiltonian or transfer matrix are Hermitian.

A natural extension of our discussion is to include more complicated boundary conditions or other representations of the Temperley-Lieb algebra [23]. Additional open problems are

the formulation of the graphical calculus for the values (2) as this would provide a more convenient formalism for computations. The main hurdle to overcome is to find a graphical rule for the reduction of the state space which has to be carried out first in order to remove non-trivial Jordan blocks in the Hamiltonian. One may also wish to extend the discussion to roots of unity other than (2). We hope to address these questions in future work [24].

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